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1989 J. Phys. A: Math. Gen. 22 5177

(http://iopscience.iop.org/0305-4470/22/24/008)

J. Phys. A: Math. Gen. 22 (1989) 5177-5186. Printed in the UK

Construction of coupling coefficients of SU(4) in a supermultiplet basis

S Ališauskas and E Norvaišas

Institute of Physics, Academy of Sciences of the Lithuanian SSR, SU-232600, Vilnius, USSR

Received 25 October 1988, in final form 26 June 1989

Abstract. Matrix elements of the unit tensor operators of ranks [1], [2] and [1, 1] in the non-orthonormal Draayer basis of SU(4) are found. The evaluation algorithms for the SU(4) coupling coefficients are proposed for coupling one arbitrary and one four-, six- or ten-dimensional irreducible representations of SU(4) in the $SU(2)\times SU(2)$ chain, as well as in the case of the semistretched coupling. The construction of the overlaps of the Draayer basis states is considered.

1. Introduction

The Clebsch-Gordan coefficients of the SU(4) group restricted to $SU(2)\times SU(2)$ play an important role in the concrete investigations in frames of the Wigner supermultiplet model. The practical algebraic expressions of the $SU(4)\supset SU(2)\times SU(2)$ isofactors are available only in some cases of the small multiplicities of the irreducible representations (irreps) of subgroups (Hecht and Pang 1969, Vladimirov 1984, Vladimirov and Gaponov 1986, Hecht *et al* 1987, Han *et al* 1989), as well as for the coupling of two symmetric irreps of SO(6) in the $SO(3)\oplus SO(3)$ basis (Norvaišas and Ališauskas 1974, Ališauskas 1983b). However, the class of isofactor needed by the supermultiplet model (see Vanagas 1988) is different from those considered by Ališauskas (1982, 1983a, 1984, 1987) and Petrauskas and Ališauskas (1987).

The coupling coefficients of an arbitrary irrep of SU(4) and an irrep with one or two squares in the Young tableau (i.e. the dimension of which is equal, respectively, to 4, 6 or 10) is of particular importance for nuclear theory in the case of one- or two-particle operators, depending on the spin-isospin variables. Norvaišas (1981) considered the isofactors of the non-orthogonal Draayer (1970) basis in the case of the second irrep with one square in the Young tableau. The use of the permutation technique for the tensor and projection operators enabled them to avoid the resubducing coefficients (transformation brackets) which are necessary in the general construction scheme of $SU(4)\supset SU(2)\times SU(2)$ isofactors (see (5.25) of Draayer (1970)) and the explicit expressions for which (see (3.1) of Norvaišas (1981) and the correction in Ališauskas (1983a)) are very complicated.

In this paper we succeeded in finding a new interpretation of some structure elements of the isofactors considered by Norvaišas (1981). Taking into account that the intrinsic (Gelfand-Zetlin) states for the Draayer basis form the double irreducible tensors of the block-diagonal subgroup $S(U(2)\oplus U(2))$ we succeeded in deriving the explicit expressions for the isofactors corresponding to the matrix elements of the SU(4) irreducible tensors of ranks [2] and [11]. For this purpose the permutation relation for the tensor and projection operators were used, which caused the appearance of the Clebsch-Gordan (CG) coefficients of the SU(2) groups of spin and isospin. Later, the action of the SU(4)-irreducible tensor operators on the standard intrinsic states was expressed either with the help of the simple matrix elements, or replaced by some SU(2)-coupled structures in frames of the spin-isospin or the block-diagonal SU(2) subgroups. In order not to complicate expressions, the dual constructions and symmetries of the coupling coefficients are used.

The expansion of the linearly dependent states of the Draayer basis (Ališauskas and Norvaišas 1979, Norvaišas and Ališauskas 1989) may be avoided in our construction only when the orthogonalisation coefficients of the dual bases are known. The overlap coefficients of the Draayer basis are necessary for orthonormalisation of the basis states by the Gram-Schmidt process or with help of the labelling operators of Moshinsky and Nagel (1963) or Partensky and Maguin (1978). (Their matrix elements in the Draayer basis for arbitrary irreps are derived by Norvaišas and Ališauskas 1989). In the appendix we discuss some alternative derivations of the closed expressions for overlaps.

2. Notation and definitions

As an alternative to Draayer (1970), the generators of the SU(2) subgroups of spin and isospin may be expressed in terms of the SU(4) generators as follows:

$$\boldsymbol{S}_+ = E_{32}+E_{14} \qquad \boldsymbol{S}_- = E_{23}+E_{41} \qquad \boldsymbol{S}_0=\tfrac{1}{2}(E_{11}-E_{22}+E_{33}-E_{44}) \tag{2.1a}$$

$$\boldsymbol{T}_+ = E_{42}+E_{13} \qquad \boldsymbol{T}_- = E_{24}+E_{31} \qquad \boldsymbol{T}_0=\tfrac{1}{2}(E_{11}-E_{22}-E_{33}+E_{44}). \tag{2.1b}$$

The irreps of SU(4) will be denoted by the Young tableau $[h_1h_2h_3h_4]$ or by $(\lambda_1\lambda_2\lambda_3)$, where $\lambda_1=h_1-h_2$, $\lambda_2=h_2-h_3$, $\lambda_3=h_3-h_4$. They correspond to the irreps $[pp'p'']$ or SO(6) where $p=\frac{1}{2}(\lambda_1+\lambda_3)+\lambda_2$, $p'=\frac{1}{2}(\lambda_1+\lambda_3)$ and $p''=\frac{1}{2}(\lambda_1-\lambda_3)$.

The basis states of the Draayer (1970) basis accept the form

$$\left|\begin{matrix}(\lambda_1\lambda_2\lambda_3)_E \\ K_SSM_S;\ K_TTM_T\end{matrix}\right\rangle = \boldsymbol{P}^S_{M_SK_S}\boldsymbol{P}^T_{M_TK_T}|\tilde{G}\{K_SK_T\}\rangle \tag{2.2}$$

where $\boldsymbol{P}^S_{M_SK_S}$, $\boldsymbol{P}^T_{M_TK_T}$ are the projection operators of the SU(2) subgroups and

$$|\tilde{G}\{K_SK_T\}\rangle = |\tilde{G}^{(\lambda_1\lambda_2\lambda_3)}_{k_1k_3}\rangle = \left|\begin{matrix} h_1 & & h_2 & & h_3 & & h_4 \\ & h_1 & & h_2 & & h_4+\frac{1}{2}\lambda_3+k_3 & \\ & & h_1 & & h_2 & & \\ & & & h_1-\frac{1}{2}\lambda_1+k_1 & & & \end{matrix}\right\rangle \tag{2.3}$$

is the special (intrinsic) Gelfand-Zetlin (GZ) state. The intrinsic GZ state (2.3) is different from those used by Draayer (1970), Ališauskas and Norvaišas (1979) and Norvaišas and Ališauskas (1989) but it may be also labelled by the same projection type parameters k_1 and k_3 of irreps $\frac{1}{2}\lambda_1$ and $\frac{1}{2}\lambda_3$ of the block-diagonal SU(2) subgroups. Here $k_1=\frac{1}{2}(K_S+K_T)$, $k_3=\frac{1}{2}(K_S-K_T)$. Inequalities (2.8) of Norvaišas and Ališauskas (1989) present (in a compact form) the restrictions of the linearly independent states of the Draayer basis. (The corresponding inequalities (11) of Ahmed and Sharp (1972) should be more specified.)

Let us introduce the operators-double tensors of the block-diagonal subgroup $S(U(2)\oplus U(2))$ in the enveloping algebra of SU(4) which are contravariant with respect to the first SU(2) subgroup and convariant with respect to the second SU(2):

$$\boldsymbol{E}^{(\overline{1/2},1/2)}_{\overline{1/2},1/2} = E_{31} \qquad \boldsymbol{E}^{(\overline{1/2},1/2)}_{-\overline{1/2},1/2} = E_{32} \qquad \boldsymbol{E}^{(\overline{1/2},1/2)}_{\overline{1/2},-1/2} = E_{41} \qquad \boldsymbol{E}^{(\overline{1/2},1/2)}_{-1/2,-1/2} = E_{42}. \tag{2.4}$$

The action of these operators on special intrinsic states is equivalent, respectively, to the action of operators of the isospin or spin $\boldsymbol{T}_-$, $\boldsymbol{S}_+$, $\boldsymbol{S}_-$ and $\boldsymbol{T}_+$. Similarly the operators-double tensors $\boldsymbol{E}^{(\bar{1},1)}_{\bar{\kappa}_1,\kappa_3}$ with the components $\boldsymbol{E}^{(\bar{1},1)}_{\bar{1},1} = \boldsymbol{E}^2_{31}$, $\boldsymbol{E}^{(\bar{1},1)}_{\bar{0},1} = \sqrt{2}E_{31}E_{32}$, etc, may be introduced. For example, the operators $\boldsymbol{E}^{(\bar{1},1)}_{\bar{1},1}$ and $\boldsymbol{E}^{(\bar{1},1)}_{\bar{0},1}$ may be replaced, respectively, by $\boldsymbol{T}^2_-$ and $\sqrt{2}(\boldsymbol{S}_+\boldsymbol{T}_- + E_{34})$ when $\boldsymbol{E}^{(\bar{1},1)}_{\bar{0},0}$ may be replaced by its eigenvalue.

We use also the correspondance between the unit (according to Racah–Biedenharn) tensor operators $\boldsymbol{a}^{(\lambda_1\lambda_2\lambda_3)}_{SM_STM_T}$ of the supermultiplet chain and the double (covariant) tensors $\boldsymbol{A}^{(j_1,j_3)}_{n_1,n_3}$ and $\boldsymbol{B}^{(j_1,j_3)}_{n_1,n_3}$ of the block-diagonal subgroup. Particularly, $\boldsymbol{a}^{(100)n_1,n_3}_{1/21/2,1/21/2}$, $\boldsymbol{a}^{(100)}_{1/2-1/2,1/2-1/2}$, $\boldsymbol{a}^{(100)}_{1/21/2,1/2-1/2}$ and $\boldsymbol{a}^{(100)}_{1/2-1/2,1/21/2}$ correspond to $\boldsymbol{A}^{(1/2,0)}_{1/2,0}$, $\boldsymbol{A}^{(1/2,0)}_{-1/2,0}$, $\boldsymbol{A}^{(0,1/2)}_{0,1/2}$ and $\boldsymbol{A}^{(0,1/2)}_{0,-1/2}$. There is also one-to-one correspondance between $\boldsymbol{a}^{(200)}_{0,-1/2}$ and $\boldsymbol{A}^{(j_1,j_3)}_{n_1,n_3}(j_1 + j_3 = 1,\ M_S = n_1 + n_3,\ M_T = n_1 - n_3)$ for at least $M_S \neq 0$ or $M_T \neq 0$, when $\boldsymbol{a}^{(200)}_{10,10} = (1/\sqrt{2})(\boldsymbol{A}^{(1,0)}_{0,0} + \boldsymbol{A}^{(0,1)}_{0,0})$ and $\boldsymbol{a}^{(200)}_{00,00} = (1/\sqrt{2})(\boldsymbol{A}^{(1,0)}_{0,0} - \boldsymbol{A}^{(0,1)}_{0,0})$. The operators $\boldsymbol{a}^{(010)}_{11,00}$, $\boldsymbol{a}^{(0,10)}_{1-1,00}$, $\boldsymbol{a}^{(010)}_{00,11}$, $\boldsymbol{a}^{(010)}_{00,1-1}$, $\boldsymbol{a}^{(010)}_{10,00}$ and $\boldsymbol{a}^{(010)}_{00,10}$ are equivalent, respectively, to $\boldsymbol{B}^{(1/2,1/2)}_{1/2,1/2}$, $-\boldsymbol{B}^{(1/2,1/2)}_{-1/2,-1/2}$, $\boldsymbol{B}^{(1/2,1/2)}_{1/2,-1/2}$, $-\boldsymbol{B}^{(1/2,1/2)}_{-1/2,1/2}$ and $(1/\sqrt{2})(\boldsymbol{B}^{(0,0)}_{0,0} \mp \tilde{\boldsymbol{B}}^{(0,0)}_{0,0})$. Here $\boldsymbol{B}^{(0,0)}_{0,0}$ corresponds to the component $x_1y_2 - x_2y_1$ and $\tilde{\boldsymbol{B}}^{(0,0)}_{0,0}$ corresponds to the component $x_3y_4 - x_4y_3$ of the antisymmetric tensor of SU(4).

Below we shall also use the notation of the decreasing factorials

$$a^{(b)} = a(a-1)\ldots(a-b+1).$$

3. Semistretched coupling of the Draayer states

Let us consider coupling of the states of the irreps $[h_1h_2h_3h_4] \times [\bar{h}_1\bar{h}_2\bar{h}_3]$ to $[h'_1h'_2h'_3h'_4]$ with the condition $h_1 + h_2 + \bar{h}_1 + \bar{h}_2 = h'_1 + h'_2$ (or $h_3 + h_4 + \bar{h}_3 = h'_3 + h'_4$). This (called semistretched) coupling of the SU(4) basis states is multiplicity-free. Special CG coefficients of the canonical (semicanonical) basis of SU(4) needed in (5.25) of Draayer (1970) turn into a product of some SU(2) CG coefficients. Analogously to the SU(3) ⊃ SO(3) case (appendix 2 of Ališauskas (1987)) it is expedient to use the inverse resubducing coefficients (the transformation brackets) which in the region typical for our problem may be expressed in terms of the expansion coefficients of the linearly dependent Draayer states.

Now the isofactors for the semistretched coupling of two orthonormal SU(4) ⊃ SU(2) × SU(2) states into the third state may be written as follows:

$$\begin{bmatrix} (\lambda_1\lambda_2\lambda_3) & (\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3) & (\lambda'_1\lambda'_2\lambda'_3) \\ \kappa; ST & \bar{\kappa}; \bar{S}\bar{T} & \kappa'; S'T' \end{bmatrix}$$
$$= \sum_{K_SK_T\bar{K}_S\bar{K}_TK'_SK'_T} \tilde{O}^{(\lambda_1\lambda_2\lambda_3;ST)}_{\kappa;K_SK_T} \tilde{O}^{(\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3;\bar{S}\bar{T})}_{\bar{\kappa};\bar{K}_S\bar{K}_T} \begin{bmatrix} S & \bar{S} & S' \\ K_S & \bar{K}_S & K'_S \end{bmatrix} \begin{bmatrix} T & \bar{T} & T' \\ K_T & \bar{K}_T & K'_T \end{bmatrix}$$
$$\times \begin{bmatrix} \lambda_1/2 & \bar{\lambda}_1/2 & \lambda'_1/2 \\ k_1 & \bar{k}_1 & k'_1 \end{bmatrix} \begin{bmatrix} \lambda_3/2 & \bar{\lambda}_3/2 & \lambda'_1/2 \\ k_3 & \bar{k}_3 & k'_3 \end{bmatrix} O^{(\lambda'_1\lambda'_2\lambda'_3;S'T')}_{K'_SK'_T;\kappa'}. \tag{3.1}$$

Here O and $\tilde{O}$ are the orthogonalisation coefficients of the projected (Draayer) and dual states; κ, $\bar{\kappa}$ and κ' are the sets of the eigenvalues of the labelling operators or some other labels of the orthonormal states. The summation parameters K'_S, K'_T

accept values corresponding to linearly independent states of the Draayer basis. The summation over all possible values of $K_S K_T$ and $\bar{K}_S \bar{K}_T$ (including the region of the linearly dependent states) allows us to escape the appearance of the expansion coefficients of the linearly dependent states of the irreps $(\lambda_1\lambda_2\lambda_3)$ and $(\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3)$.

The inverse orthogonalisation coefficients $\tilde{O}$ coincide with the boundary resubducing coefficients. The explicit expressions for O and $\tilde{O}$ for the states of two-parametric irreps $(\lambda_1\lambda_2 0)$ and $(0\lambda_2\lambda_3)$ will be given in a future publication. For irreps $(\lambda_1 00)$, $(00\lambda_3)$ or $(0\lambda_2 0)$ the coefficients $\tilde{O}$ are equivalent, respectively, to $C^{\lambda_1}_{k_1 S}$, $(-1)^{S+k_3} C^{\lambda_3}_{k_3 S}$ (in the both cases $S=T$) or $\tilde{C}^{\lambda_2}_{ST}$ (see (A2*a*) and (A2*b*) of Norvaišas and Ališauskas 1989). For special irreps (100), (200) and (010) these factors are equal to ± 1 or $\pm 1/\sqrt{2}$.

4. Action of the unit tensor operators into the Draayer basis

The SU(4)-reduced matrix elements of the unit (according to Racah–Biedenharn) tensor operators are equal to 1 for the given values of shifts $\Delta_i = h'_i - h_i$ $(i=1,2,3,4)$; the complete (usual) matrix elements in the orthogonal basis coincide with the coupling coefficients of SU(4). We use the permutation relation

$$\boldsymbol{a}^{(\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3)}_{\bar{S}\bar{M}_S\bar{T}\bar{M}_T} \boldsymbol{P}^{S}_{M_S K_S} \boldsymbol{P}^{T}_{M_T K_T} = \sum_{S'T'm_S m_T K''_S K''_T} \frac{(2S+1)(2T+1)}{(2S'+1)(2T'+1)} \begin{bmatrix} S & \bar{S} & S' \\ M_S & \bar{M}_S & M'_S \end{bmatrix} \begin{bmatrix} T & \bar{T} & T' \\ M_T & \bar{M}_T & M'_T \end{bmatrix}$$
$$\times \begin{bmatrix} S & \bar{S} & S' \\ K_S & m_S & K''_S \end{bmatrix} \begin{bmatrix} T & \bar{T} & T' \\ K_T & m_T & K''_T \end{bmatrix} \boldsymbol{P}^{S'}_{M'_S K''_S} \boldsymbol{P}^{T'}_{M'_T K''_T} \boldsymbol{a}^{(\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3)}_{\bar{S} m_S \bar{T} m_T} \tag{4.1}$$

when acting with the unit tensor operator $\boldsymbol{a}^{(\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3)}_{\bar{S}\bar{M}_S\bar{T}\bar{M}_T}$ on the Draayer basis states (2.2).

Now, the action of the operators $\boldsymbol{a}^{(\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3)}_{\bar{S} m_S \bar{T} m_T}$ (equivalent to the double tensors $(A^{(j_1,j_3)}_{n_1,n_3}$ or $\boldsymbol{B}^{(j_1,j_3)}_{n_1,n_3}$ of the block-diagonal subgroup $U(2)\oplus U(2)$) on the intrinsic states (2.3) gives either the expansion immediately in terms of the standard intrinsic states, or the less specific GZ states which need to be replaced by structures which include operators of type (2.4) acting on standard intrinsic states. In the first case their matrix elements, i.e. special coupling coefficients of SU(4) (see Baird and Biedenharn 1963, Biedenharn and Louck 1968, Le Blanc and Hecht 1987, Ališauskas *et al* 1972) are equivalent to the CG coefficients of SU(2) with some dimension-type factors. In the other case, the corresponding structures are particularly determined by the coupling in frames of the block-diagonal SU(2) subgroups and by the coupling coefficients and the matrix elements of the SU(4) generators. The terms of the third kind may include both constructions.

The operators $\boldsymbol{A}^{(j_1,0)}_{n_1,0}$ and $\boldsymbol{B}^{(0,0)}_{0,0}$ act completely in frames of the first block-diagonal SU(2) and give only the terms that correspond to the semistretched coupling:

$$\boldsymbol{A}^{(j_1 0)}_{n_1 0} |\tilde{G}^{(\lambda_1\lambda_2\lambda_3)}_{k_1 k_3}\rangle = \delta_{\Delta_2 0}\delta_{\Delta_4 0} \sum_{\lambda'_1\lambda'_2} \begin{bmatrix} \lambda_1/2 & j'_1 & \lambda'_1/2 \\ k_1 & n_1 & k'_1 \end{bmatrix} |\tilde{G}^{(\lambda'_1\lambda'_2\lambda_3)}_{k'_1 k_3}\rangle. \tag{4.2}$$

Only the operators $\boldsymbol{A}^{(0,j_3)}_{0,n_3}$ and $\tilde{\boldsymbol{B}}^{(0,0)}_{0,0}$ acting on (2.3) give the antisemistretched terms (with $\Delta_1 = \Delta_2 = 0$ if $\bar{\lambda}_3 = 0$):

$$\delta_{\Delta_1 0}\delta_{\Delta_2 0} \sum_{\lambda'_2\lambda'_3} \left[\frac{(h_1-h'_3+2)(h_2-h'_3+1)(h_1-h'_4+3)(h_2-h'_4+2)}{(h_1-h_3+2)(h_2-h_3+1)(h_1-h_4+3)(h_2-h_4+2)}\right]^{1/2}$$
$$\times \begin{bmatrix} \lambda_3/2 & j_3 & \lambda'_3/2 \\ k_3 & n_3 & k'_3 \end{bmatrix} |\tilde{G}^{(\lambda_1\lambda'_2\lambda'_3)}_{k_1 k'_3}\rangle. \tag{4.3}$$

(In the case of the operators $\boldsymbol{B}_{0,0}^{(0,0)}$ and $\tilde{\boldsymbol{B}}_{0,0}^{(0,0)}$ the CG coefficients of SU(2) in (4.2) and (4.3) are trivial.)

In addition, the action of $\boldsymbol{A}_{0,n_3}^{(0,1/2)}$ on (2.3) gives the semistretched terms:

$$\delta_{\Delta_3 0}\delta_{\Delta_4 0}\sum_{\lambda_1'\lambda_2'} F_{(01/2)}^{[\Delta_1\Delta_2 00]} \times \sum_{\kappa_1\kappa_3 k_1'} D_{01/2;0n_3,\kappa_3}^{[\Delta_1\Delta_2 00]}(k_3)\begin{bmatrix}\lambda_1/2 & \frac{1}{2} & \lambda_1'/2\\ k_1 & \kappa_1 & k_1'\end{bmatrix} E_{\bar{\kappa}_1,\kappa_3}^{(\bar{1}/2,1/2)}|\tilde{G}_{k_1',k_3+n_3+\kappa_3}^{(\lambda_1'\lambda_2'\lambda_3)}\rangle \tag{4.4}$$

where $\Delta_1+\Delta_2=1$,

$$F_{(01/2)}^{[\Delta_1\Delta_2 00]}=[(h_1'-h_3+1)^{\Delta_1}(h_1'-h_4+2)^{\Delta_1}(h_2'-h_3)^{\Delta_2}(h_2'-h_4+1)^{\Delta_2}]^{-1} \tag{4.5}$$

$$\begin{aligned} D_{01/2;0-1/2;-1/2}^{[\Delta_1\Delta_2 00]}(k_3) &= D_{01/2;01/2,1/2}^{[\Delta_1\Delta_2 00]}(-k_3)\\ &=(h_1+1)\Delta_1+h_2\Delta_2-(h_3+h_4)/2-k_3+2 \end{aligned} \tag{4.6a}$$

$$\begin{aligned} D_{01/2,0-1/2;1/2}^{[\Delta_1\Delta_2 00]}(k_3) &= D_{01/2;01/2;-1/2}^{[\Delta_1\Delta_2 00]}(-k_3)\\ &=[(\lambda_3/2-k_3+1)(\lambda_3/2+k_3)]^{1/2}. \end{aligned} \tag{4.6b}$$

Of course, the technique of section 3 is more convenient in the semistretched case when (4.3) together with (4.1) is sufficient in the antisemistretched case. The terms neither semistretched nor antisemistretched may appear in the case of the operators $\boldsymbol{a}^{(200)}$ or $\boldsymbol{a}^{(010)}$. Particularly, the action of $\boldsymbol{A}_{n_1,n_3}^{(1/2,1/2)}$ on (2.3) gives

$$(\delta_{\Delta_1 1}+\delta_{\Delta_2 1})(\delta_{\Delta_3 1}+\delta_{\Delta_4 1})\sum_{\lambda_1'\lambda_2'\lambda_3'} F_{(1/21/2)}^{[\Delta_1\Delta_2\Delta_3\Delta_4]}\begin{bmatrix}\lambda_1/2 & 1/2 & \lambda_3'/2\\ k_1 & n_1 & k_1'\end{bmatrix} \times\begin{bmatrix}\lambda_3/2 & 1/2 & \lambda_3'/2\\ k_3 & n_3 & k_3'\end{bmatrix}|\tilde{G}_{k_1'k_3'}^{(\lambda_1'\lambda_2'\lambda_3')}\rangle \tag{4.7}$$

where

$$F_{(1/21/2)}^{[\Delta_1\Delta_2\Delta_3\Delta_4]}=\left[\frac{(h_\alpha-h_\beta-\alpha+\beta-1)(h_{\bar{\alpha}}-h_\beta-\bar{\alpha}+\beta-1)}{(h_\alpha-h_\beta-\alpha+\beta+1)(h_{\bar{\alpha}}-h_\beta-\bar{\alpha}+\beta)}\right]^{1/2}. \tag{4.8}$$

Here and below $\alpha=1$ or 2, $\bar{\alpha}=2$ or l $(\alpha\neq\bar{\alpha})$; $\beta=3$ or 4, $\bar{\beta}=4$ or 3 $(\beta\neq\bar{\beta})$ so that $\Delta_\alpha=\Delta_\beta=1$, $\Delta_{\bar{\alpha}}=\Delta_{\bar{\beta}}=0$.

The action of $\boldsymbol{A}_{0,n_3}^{(0,1)}$ on (2.3) gives

$$(\delta_{\Delta_1 1}+\delta_{\Delta_2 1})(\delta_{\Delta_3 1}+\delta_{\Delta_4 1})(\lambda_3+1)^{-1/2}\sum_{\lambda_1'\lambda_2'\lambda_3'} F_{(01)}^{[\Delta_1\Delta_2\Delta_3\Delta_4]}\sum_{\kappa_1\kappa_3 k_1'} D_{01;0n_3;\kappa_3}^{[\Delta_1\Delta_2\Delta_3\Delta_4]}(k_3) \times\begin{bmatrix}\lambda_1/2 & 1/2 & \lambda_1'/2\\ k_1 & \kappa_1 & k_1'\end{bmatrix} E_{\bar{\kappa}_1,\kappa_3}^{(\bar{1}/2,1/2)}|\tilde{G}_{k_1';k_3+n_3+\kappa_3}^{(\lambda_1'\lambda_2'\lambda_3')}\rangle \tag{4.9}$$

$$F_{(01)}^{[\Delta_1\Delta_2\Delta_3\Delta_4]}=(h_\alpha-h_{\bar{\beta}}-\alpha+\bar{\beta})^{-1} \times\left[\frac{h_{\bar{\alpha}}-h_\beta-\bar{\alpha}+\beta-1}{(h_\alpha-h_\beta-\alpha+\beta-1)(h_\alpha-h_\beta-\alpha+\beta+1)(h_{\bar{\alpha}}-h_\beta-\bar{\alpha}+\beta)}\right]^{1/2} \tag{4.10}$$

$$\begin{aligned} D_{01;0-1;-1/2}^{[\Delta_1\Delta_2\Delta_3\Delta_4]}(k_3) &= D_{01;01;1/2}^{[\Delta_1\Delta_2\Delta_3\Delta_4]}(-k_3)\\ &=\sqrt{2}[(h_1+1)\Delta_1+h_2\Delta_2-(h_3+h_4)/2-k_3+2]\\ &\quad\times[\lambda_3/2+(-1)^{\Delta_3}k_3+\Delta_3]^{1/2} \end{aligned} \tag{4.11a}$$

$$\begin{aligned} D_{01;0-1;1/2}^{[\Delta_1\Delta_2\Delta_3\Delta_4]}(k_3) &= D_{01;01;-1/2}^{[\Delta_1\Delta_2\Delta_3\Delta_4]}(-k_3)\\ &=[2(\lambda_3/2-k_3+\Delta_3+1)^{(\Delta_3+1)}(\lambda_3/2+k_3)^{(\Delta_4+1)}]^{1/2} \end{aligned} \tag{4.11b}$$

$$D^{[\Delta_1\Delta_2\Delta_3\Delta_4]}_{01;00;1/2}(k_3)$$
$$= (-1)^{\Delta_4} D^{[\Delta_1\Delta_2\Delta_3\Delta_4]}_{01;00;-1/2}(-k_3)$$
$$= [(h_1+1)\Delta_1 + h_2\Delta_2 - (h_3+1)\Delta_3 - h_4\Delta_4 + 2k_3 + 2]$$
$$\times [\lambda_3/2 + (-1)^{\Delta_3} k_3 + \Delta_3]. \qquad (4.11c)$$

The action of $\boldsymbol{B}^{(1/2,1/2)}_{n_1,n_3}$ on (2.3) gives

$$(\delta_{\Delta_1 1} + \delta_{\Delta_2 1})(\delta_{\Delta_3 1} + \delta_{\Delta_4 1}) \sum_{\lambda'_1\lambda'_2\lambda'_3} \left(\frac{h_{\bar{\alpha}} - h_\beta - \bar{\alpha} + \beta - 1}{h_{\bar{\alpha}} - h_\beta - \bar{\alpha} + \beta} \right)^{1/2}$$
$$\times \begin{bmatrix} \lambda_1/2 & 1/2 & \lambda'_1/2 \\ k_1 & n_1 & k'_1 \end{bmatrix} \begin{bmatrix} \lambda_3/2 & 1/2 & \lambda'_3/2 \\ k_3 & n_3 & k'_3 \end{bmatrix} |\tilde{G}^{(\lambda'_1\lambda'_2\lambda'_3)}_{k'_1k'_3}\rangle \qquad (4.12)$$

and the action of $\tilde{\boldsymbol{B}}^{(0,0)}_{0,0}$ on (2.3) gives

$$(\delta_{\Delta_1 1} + \delta_{\Delta_2 1})(\delta_{\Delta_3 1} + \delta_{\Delta_4 1}) \sum_{\lambda'_1\lambda'_2\lambda'_3} F^{[\Delta_1\Delta_2\Delta_3\Delta_4]}_{(00)} \sum_{\kappa_1\kappa_3k'_1k'_3} \begin{bmatrix} \lambda_1/2 & 1/2 & \lambda'_1/2 \\ k_1 & \kappa_1 & k'_1 \end{bmatrix}$$
$$\times \begin{bmatrix} \lambda'_3/2 & 1/2 & \lambda_3/2 \\ k'_3 & \kappa_3 & k_3 \end{bmatrix} \boldsymbol{E}^{(1/2,1/2)}_{\bar{\kappa}_1,\kappa_3} |\tilde{G}^{(\lambda'_1\lambda'_2\lambda'_3)}_{k'_1k'_3}\rangle \qquad (4.13)$$

where

$$F^{[\Delta_1\Delta_2\Delta_3\Delta_4]}_{(00)} = \frac{(-1)^{\Delta_3}}{(h_\alpha - h_{\bar{\beta}} - \alpha + \bar{\beta})} \left[\frac{\lambda_3^{\Delta_4}(\lambda_3+2)^{\Delta_3}(h_{\bar{\alpha}} - h_\beta - \bar{\alpha} + \beta - 1)}{(\lambda_3+1)(h_{\bar{\alpha}} - h_\beta - \bar{\alpha} + \beta)} \right]^{1/2}. \qquad (4.14)$$

We have also found semistretched terms that are too cumbersome to be presented here.

In the next step the operators $\boldsymbol{E}^{(1/2,1/2)}_{\bar{\kappa}_1,\kappa_3}$ see (2.4)) in (4.4), (4.9) and (4.13) should be replaced by the operators $\boldsymbol{S}_\pm$ or $\boldsymbol{T}_\pm$ that may be included into the structure of the projection operators of (4.1).

In this way the operators $\boldsymbol{a}^{(\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3)}_{\bar{S}\bar{M}_S\bar{T}\bar{M}_+}$ may be represented in the Draayer basis. Depending on the increase of S, T and decrease of λ_2, the parameters K'_S, K'_T which appear may be shifted into the region of the linearly dependent states. The expansion coefficients orthogonalised with the help of $O^{(\lambda_1\lambda_2\lambda_3;ST)}_{K_SK_T;\kappa}$ and $\tilde{O}^{(\lambda'_1\lambda'_2\lambda'_3,S'T')}_{\kappa';K'_SK'_T}$ give (in a factorised form) the coupling coefficients of SU(4) ⊃ SU(2) × SU(2).

5. Concluding remarks

In this paper we presented the algorithms for evaluating the coupling coefficients (unit tensor operators) of SU(4) ⊃ SU(2) × SU(2) (SO(6) ⊃ SO(3) ⊕ SO(3)) in the non-orthonormal Draayer basis or in its dual basis. All their structure elements are tabulated in an algebraic-polynomial form or expressed in the framework of the theory of angular momentum (thus they are also tabulated). When the general approach (see section 4) leads to the most complicated construction an alternative presented in section 3 is possible. However, for a recursive construction of the more general SU(4) unit tensor operators in the Draayer basis only the approach of section 4 is convenient. The remaining problem is the orthonormalisation of isofactors, for which the overlaps of the Draayer states are needed. The corresponding orthonormal isofactors of SU(4) ⊃ SU(2) × SU(2) coincide with the corresponding isofactors of $S_N \supset S_{N'} \oplus S_{N''}$ (see Vanagas 1971, Haase and Butler 1984) with $N''=1$ or 2.

Appendix. On the overlaps of the Draayer basis states

The most convenient expression for the overlaps of irreps $[h_1h_200]$ or $(\lambda_1\lambda_20)$ is (3.6) of Ališauskas (1982) (see also (5.4) of Ališauskas 1984, where $L_1=S+T$, $L_2=|S-T|$, $\lambda=\lambda_1$, $\nu=\lambda_2$, $\Delta_0=0$ or 1 and $\nu-L_2-\Delta_0$ is even). The number of terms in each sum does not exceed the multiplicity of S, T in $(\lambda_1\lambda_20)$. The case with $T>S$ (more interesting from the physical point of view) may be obtained by the substitution $S\leftrightarrow T$.

Equation (3.1) with $\bar\lambda_1=\bar\lambda_2=\lambda_3=0$, $\lambda'_1=\lambda_1$, $\lambda'_2=\lambda_2$, $\lambda'_3=\lambda_3$ allows one to write the recursive expression for overlaps:

$$\left\langle\begin{matrix}(\lambda_1\lambda_2\lambda_3)_E\\ K_SS;\,K_TT\end{matrix}\right|\left.\begin{matrix}(\lambda_1\lambda_2\lambda_3)_E\\ K'_SS;\,K'_TT\end{matrix}\right\rangle
=(-1)^{k_3-k'_3}[(\tfrac12\lambda_3-k_3)!(\tfrac12\lambda_3+k_3)!(\tfrac12\lambda_3-k'_3)!(\tfrac12\lambda_3+k'_3)!]^{1/2}$$
$$\times\sum_{S_0T_0s_3}\begin{bmatrix}S_0 & s_3 & S\\ k_1 & k_3 & K_S\end{bmatrix}\begin{bmatrix}T_0 & s_3 & T\\ k_1 & -k_3 & K_T\end{bmatrix}\begin{bmatrix}S_0 & s_3 & S\\ k'_1 & k'_3 & K'_S\end{bmatrix}\begin{bmatrix}T_0 & s_3 & T\\ k'_1 & -k'_3 & K'_T\end{bmatrix}$$
$$\times\frac{2s_3+1}{(\tfrac12\lambda_3-s_3)!(\tfrac12\lambda_3+s_3+1)!}\left\langle\begin{matrix}(\lambda_1\lambda_20)_E\\ k_1,\,S_0T_0\end{matrix}\right|\left.\begin{matrix}(\lambda_1\lambda_20)_E\\ k'_1,\,S_0T_0\end{matrix}\right\rangle. \tag{A1}$$

(Here and below we omit the parameters M_s, M_T.)

The application of the intrinsic states (2.3) allows us a more direct derivation of the expressions for overlaps, related to (3.2) of Ališauskas (1983a). For this purpose we use the projection operators of Löwdin (1964) and Shapiro (1965) expanded as an ordered polynomial in $\boldsymbol{S}_-$, $\boldsymbol{T}_+$, $\boldsymbol{T}_-$, $\boldsymbol{S}_+$ and the permutation formulas (Asherova and Smirnov 1968) of the factorised SU(4) generators together with the specific action of the generators of the spin, isospin and block-diagonal subgroup on the intrinsic states. The overlap has been expanded in terms of the matrix elements of the factorised SU(3) generators $E_{13}^xE_{23}^{a-x}E_{32}^{a-x'}E_{31}^{x'}$ between the semi-highest weight states of SU(3) (that are obtained from the highest weight states by acting with the SU(2) generators). The obtained Saalschutzian $_4F_3(1)$ series (see Slater 1966) have appeared after applying the corrected equation (2.11) of Asherova and Smirnov (1968) with $+b-a$ inserted before $-t$ on the right-hand side. In this way the overlap is expressed as

$$\left\langle\begin{matrix}(\lambda_1\lambda_2\lambda_3)_E\\ K_SS;\,K_TT\end{matrix}\right|\left.\begin{matrix}(\lambda_1\lambda_2\lambda_3)_E\\ K'_SS;\,K'_TT\end{matrix}\right\rangle
=(2S+1)(2T+1)\left[\frac{(T-K_T)!(T-K'_T)!(S+K_s)!(S+K'_S)!}{(T+K_T)!(T+K'_T)!(S-K_S)!(S-K'_S)!}\right.$$
$$\left.\times\frac{(\tfrac12\lambda_1+k_1)!(\tfrac12\lambda_1+k'_1)!(\tfrac12\lambda_3-k_3)!(\tfrac12\lambda_3-k'_3)!}{(\tfrac12\lambda_1-k_1)!(\tfrac12\lambda_1-k'_1)!(\tfrac12\lambda_3+k_3)!(\tfrac12\lambda_3+k'_3)!}\right]^{1/2}$$
$$\times\sum_{typ}\frac{(-1)^t(T+K_T+t)!(T+K'_T+t)!(\tfrac12\lambda_3+k'_3+y)!}{t!(2T+t+1)!y!(\tfrac12\lambda_3-k'_3-y)!(k'_3-k_3+y)!}$$
$$\times\frac{(\tfrac12\lambda_1-k_1+p)![\tfrac12(\lambda_1+\lambda_3)+\lambda_2+k_1-k'_3-y-p]!}{[\tfrac12(\lambda_1+\lambda_3)+\lambda_2-T-t]!p!(k'_1-k_1+p)!(\tfrac12\lambda_1+k_1-p)!}$$
$$\times\frac{1}{(T+k_1-k'_3+t-y-p)!}\sum_r\frac{(-1)^r(S-K_s+r)!(S-K'_s+r)!}{r!(2S+r+1)!(S-K'_s-y+r-p)!}$$
$$\times\frac{[\tfrac12(\lambda_1+\lambda_3)+\lambda_2-T-k_1-k'_1-t]!}{[\tfrac12(\lambda_1+\lambda_3)+\lambda_2-S-T-k_1+k'_3-r-t+y+p]!} \tag{A2}$$

where the sum over r (together with the subsequent factor) may be expressed as

$$\frac{(S-K_S)!(S-K'_S)!(-1)^{K_S-K'_S+y+p}}{[\frac{1}{2}(\lambda_1+\lambda_3)+\lambda_2+S-T-k_1+k'_3+y+p+1]!}$$
$$\times\sum_z \frac{(-1)^z(y+p+z)![\frac{1}{2}(\lambda_1+\lambda_3)+\lambda_2+S-T-k_1+k'_3-t-z]!}{z!(S-K_S-z)!(S+K'_S-z)!(K_S-K'_S+z)!}. \tag{A3}$$

In addition the following inequality should be satisfied here by the summation parameters:

$$(\lambda_1+\lambda_3)/2+\lambda_2-S-T-k_1+k'_3-t+y+p\geqslant 0.$$

Expression (A2) or (A2), (A3) is indefinite unless $k_1+k'_1\leqslant 0$ and $k_3+k'_3\geqslant 0$. The phase relation (2.9) of Norvaišas and Ališauskas (1989) (applied together or separately to K_SK_T and $K'_SK'_T$) allows us to use (A2) for $k_1+k'_1\geqslant 0$, $k_3+k'_3\leqslant 0$ or $k_1\geqslant k'_1$, $k_3\leqslant k'_3$, when the substitution $S\leftrightarrow T$, $K_S\leftrightarrow -K_T$, $K'_S\leftrightarrow -K'_T$ allows us to escape the indefiniteness for $k_1+k'_1\geqslant 0$, $k_3+k'_3\geqslant 0$.

The overlaps do not change after the transposition $S\leftrightarrow T$, $K_S\leftrightarrow K_T$, $K'_S\leftrightarrow K'_T$. The phase factor $(-1)^{K_T-K'_T}$ appears after the transposition $\lambda_1\leftrightarrow\lambda_3$ applied together with the reflection of K_S and K'_S.

Our expression (A2), (A3) includes finite intervals of summation for the fixed parameters λ_1, λ_3, S and λ_2-T. Therefore, it is convenient for small values of S and large (near to λ_2) values of T. Equation (A1) or rather complicated transformations of (A2), (A3) allow us to find the polynomial in λ_2, S, T expressions of overlaps for small values of λ_1, λ_3. Some examples that are presented below.

The overlaps of dual states of the basis $\bar{E}$ in the case of the irreps $(\lambda_1\lambda_2 0)$ were found in Petrauskas and Ališauskas (1987). By analogy with (5.13) of Ališauskas (1987)†, the analytical inversion (substitution of λ_1, λ_2, λ_3, S, T by $-\lambda_1-2$, $-\lambda_2-2$, $-\lambda_3-2$, $-S-1$, $-T-1$ applied to (A2) together with the factor

$$\frac{(\lambda_2+1)(\lambda_1+\lambda_2+2)(\lambda_2+\lambda_3+2)(\lambda_1+\lambda_2+\lambda_3+3)}{2(2S+1)^2(2T+1)^2} \tag{A4}$$

gives the overlaps of the $\bar{E}$ states. This result may be convenient especially in the non-overcomplete cases when only the phase factor (see (2.9) of Norvaišas and Ališauskas 1989) remains in the expansion coefficients R (see (A11) of Norvaišas and Ališauskas (1989)).

The inverse orders of the Gram–Schmidt procedure should be chosen in the case of the dual bases E and $\bar{E}$. The choice of these orders requires additional investigation unless $\lambda_1=0$, or $\lambda_2=0$, or $\lambda_3=0$, when the labels are linearly ordered. Since the explicit orthogonalisation coefficients for the Draayer and dual states of the irreps $(\lambda_1\lambda_2 0)$ will be given in a future publication, we present here the concrete expressions the overlaps for irreps $(1\lambda_2 1)$ and $(2\lambda_2 1)$:

$$\left\langle \begin{matrix}(1\lambda_2 1)_E\\ K_SS;\,K_TT\end{matrix}\middle| \begin{matrix}(1\lambda_2 1)_E\\ K'_SS;\,K'_TT\end{matrix}\right\rangle$$
$$=\frac{(2S+1)(2T+1)}{(\lambda_2-S-T-\Delta_0+2)!!}$$

† The first multiplication symbol × on the right-hand side of (5.11) of Ališauskas (1987) should be corrected to a plus sign, +. The corresponding parameter $|K'|\geqslant\Delta+\delta$.

$$\times \frac{\lambda_2!(\lambda_2+2)!}{(\lambda_2-S+T-\Delta_0+3)!!(\lambda_2+S-T-\Delta_0+3)!!(\lambda_2+S+T-\Delta_0+4)!!}$$

$$\times\{\delta_{K_S K'_S}\delta_{K_T K'_T}[(\lambda_2+3)[(\lambda_2+3)^2-S(S+1)-T(T+1)-1]$$

$$+(K_S-K_T)(T-S)(S+T+1)]^{1-\Delta_0}(\lambda_2+2)^{\Delta_0}$$

$$-\delta_{\Delta_0 0}\delta_{K_S K'_T}\delta_{K_T K'_S}2(\lambda_2+3)[S(S+1)T(T+1)]^{1/2}\} \qquad \text{(A5)}$$

(here K_S, $K_T = 1$, 0 or 0, 1),

$$\left\langle \begin{matrix} (2\lambda_2 1)_E \\ K_S S;\ K_T T \end{matrix} \middle| \begin{matrix} (2\lambda_2 1)_E \\ K'_S S;\ K'_T T \end{matrix} \right\rangle$$

$$= \frac{(2S+1)(2T+1)}{4(\lambda_2-S-T-\Delta_0+3)!!}$$

$$\times \frac{\lambda_2!(\lambda_2+3)!}{(\lambda_2-S+T+\Delta_0+3)!!(\lambda_2+S-T+\Delta_0+3)!!(\lambda_2+S+T-\Delta_0+5)!!}$$

$$\times\{\delta_{K_S 3/2}\delta_{K_T 1/2}\delta_{K'_S 3/2}\delta_{K'_T 1/2}[(\lambda_2+4)[4(\lambda_2+4)^{(2)}-(2S+1)^2-4]$$

$$-(\lambda_2+2)(2T+1)[2T+1+(-1)^{\Delta_0}(2S+1)]]$$

$$+\delta_{K_S,-1/2}\delta_{K_T 1/2}\delta_{K'_S,-1/2}\delta_{K'_T 1/2}[4(\lambda_2+4)^{(3)}-(\lambda_2+3)(2S+1)^2$$

$$-(\lambda_2+3)(2T+1)^2+2(-1)^{\Delta_0}(2S+1)(2T+1)]$$

$$-\delta_{K_S 3/2}\delta_{K_T 1/2}\delta_{K'_S 1/2}\delta_{K'_T 3/2}(\lambda_2+4)[(2S-1)(2S+3)(2T-1)(2T+3)]^{1/2}$$

$$+\delta_{K_S 3/2}\delta_{K_T 1/2}\delta_{K'_S,-1/2}\delta'_{K'_T,-1/2}[2S+1-(-1)^{\Delta_0}(\lambda_2+3)(2T+1)]$$

$$\times[2(2S-1)(2S+3)]^{1/2}$$

$$+\text{terms related by symmetries}\}. \qquad \text{(A6)}$$

Expressions (A5) and (A6) have been derived from (A1) by a cumbersome but straightforward evaluation when expressions (A2) helped us to make conjectures about expansion into the indecomposable polynomials. Equation (A1) allowed us to obtain simple expressions for overlaps in the cases enumerated by (5.15) of Norvaišas and締Ališauskas (1989).

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